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The diffraction of plane elastic unsteady waves by a delaminated inclusion in the case of smooth contact in the delamination region☆

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ABSTRACT

A solution of the problem of the diffraction of unsteady elastic waves by a thin strip-like delaminated rigid inclusion in an unbounded elastic medium under conditions of planer strain is proposed. We have in mind an inclusion, one side of which is completely bonded with the medium while, the other side is delaminated and conditions of smooth contact are satisfied on it. The method of solution is based on the use of discontinuous solutions of the Lamé equations of motion under conditions of planer strain, which have been constructed earlier in the space of Laplace transforms. As a result, the problem reduces to solving a system of three singular integral equations for the transforms of the unknown discontinuities. The inverse transforms are found by a numerical method, based on the replacement of a Mellin integral by a Fourier series.

Similar problems have been considered earlier in a static formulation [1,2] and for the case of harmonic oscillations.[3]

1. Formulation of the problem

Suppose a thin rigid inclusion, located in the xy plane in the domain $|y| \leq h$; $|x| \leq a$, is situated in an elastic body (the matrix) which is under conditions of planer strain. Its lower side $y = -h/2$ is completely bonded to the elastic medium while the side $y = h/2$ is delaminated and conditions of smooth contact are satisfied on it. Suppose a plane longitudinal wave with a potential $\varphi_0(x,y,t)$ or a plane transverse shear wave with a potential $\psi_0(x,y,t)$ acts on the inclusion at the instant $t = 0$.

We will denote the displacements caused by these waves by $u_0(x,y,t)$ and $\upsilon_0(x,y,t)$. The displacements in the matrix can then be represented in the form of the sum of two terms:

$$u = u_0 + u_1, \quad \upsilon = \upsilon_0 + \upsilon_1$$

where u_1 and υ_1 are the displacements in the matrix caused by the waves reflected
from the inclusion. These displacements satisfy the Lamé equations of motion for plane strain and zero initial conditions when $t = 0$.

In view of the small thickness of the inclusion, we will formulate the boundary conditions as viewed from the external medium into the inclusion relative to its middle plane. In the case when one side of the inclusion is completely bonded and the other side is under conditions of smooth contact, the stresses and strains in its middle plane undergo discontinuities, the jumps of which we will denote as follows:

$$\{\sigma_y^1\} = \chi_1(x,t), \quad \{\tau_{yx}^1\} = \chi_2(x,t), \quad \{\upsilon\} = 0, \quad \{u\} = \chi_4(x,t), \quad |x| \leq a \tag{1.1}$$

where $\{\sigma_y^1\}$ and $\{\tau_{yx}^1\}$ are the stresses in the matrix caused by the reflected waves, and the notation

$$\{f\} = f(x,+0,t) - f(x,-0,t)$$

is adopted. It follows from the conditions for the joining of the medium that

$$\chi_4(\pm a, t) = 0 \tag{1.2}$$

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By virtue of the rigidity of the inclusion, the equalities

$$\tau_{yx}^1(x,+0,t) = -\tau_{yx}^0(x,0,t),\quad \upsilon_1(x,-0,t) = a_1(t)+\gamma(t)x-\upsilon_0(x,0,t)$$
$$u_1(x,-0,t) = a_2(t)-u_0(x,0,t),\quad |x|\le a \tag{1.3}$$

follow from the complete bonding and smooth contact conditions, where $\{\tau_{yx}^0\}$ are the sheat stresses in the incident wave field, $a_1(t)$ and $a_2(t)$ are the unknown displacements of the inclusion along the y and x axes and $\gamma(t)$ is the unknown angle of rotation of the inclusion, which are determined from the equations of motion of the inclusion as a rigid body:

$$m\ddot{a}_j(t) = R_j(t),\quad j = 1,2,\quad J\ddot{\gamma}(t) = L(t)$$
$$R_j(t) = \int_{-a}^{a}\chi_j(x,t)dx,\quad L(t) = \int_{-a}^{a}\chi_1(x,t)xdx,\quad m = 2a\rho_0 h,\quad J = \frac{4}{3}ma^2 \tag{1.4}$$

where m is the per unit length mass of the inclusion, J is the moment of inertia of unit length of the inclusion, ρ_0 is the density of the inclusion, and $R_j(t)$ and $L(t)$ are the reactive forces and moment on the inclusion on the part of the matrix.

2. Method of solution

In order to solve the initial boundary-value problem formulated above, we will first use an integral Laplace transformation with respect to time (p is the transform parameter). Then, from relations (1.3) for the transforms of the corresponding functions $V_1, V_0, U_1, U_0, T_{yx}^1, T_{yx}^0, G, \bar{L}, A_j, \bar{R}_j, X_j (j = 1, 2)$, it follows that

$$\mathrm{T}_{yx}^1(x,+0,p) = -\mathrm{T}_{yx}^0(x,0,p)$$
$$V_1(x,-0,p) = A_1(p)+G(p)x-V_0(x,0,p),\ \ U_1(x,-0,p) = A_2(p)-U_0(x,0,p);\ |x|\le a \tag{2.1}$$

After the Laplace transformation, the equations of motion of the inclusion (1.4) take the form

$$mp^2A_j(p) = \bar{R}_j(p),\quad j = 1,2,\quad Jp^2G(p) = \bar{L}(p)$$
$$\bar{R}_j(p) = \int_{-a}^{a}\mathrm{X}_j(x,p)dx,\quad \bar{L}(p) = \int_{-a}^{a}\mathrm{X}_1(x,p)xdx \tag{2.2}$$

We will represent the transforms of the displacements and stresses in the matrix, caused by the reflected wave, in the form of the discontinuous solution of the Laplace-transformed Lamé equations. For this purpose, one must put $\kappa_j = i_p/c_j (j = 1,2)$ in the corresponding formula for the discontinuous solution of the Lamé equations of motion in the case of harmonic oscillations (Ref. 4, formula (1.2)), where c_1 and c_2 are the velocities of the longitudinal and transverse waves in the matrix. As a result, the required transforms of the displacements and shear stress are given by the formulae

$$U_1 = \frac{1}{\mu p_2^2}\int_{-a}^{a}\mathrm{X}_1(\eta,p)\frac{\partial^2}{\partial x\partial y}(K_1-K_2)d\eta+\frac{1}{\mu p_2^2}\int_{-a}^{a}\mathrm{X}_2(\eta,p)\tilde{K}_{12}d\eta$$
$$V_1 = \frac{1}{\mu p_2^2}\int_{-a}^{a}\mathrm{X}_1(\eta,p)\tilde{K}_{12}d\eta+\frac{1}{p_2^2}\int_{-a}^{a}\mathrm{X}_4'(\eta,p)\hat{K}_{12}d\eta$$
$$\mathrm{T}_{yx}^1 = \frac{1}{p_2^2}\int_{-a}^{a}\mathrm{X}_1(\eta,p)K_{2\hat{1}}d\eta+$$
$$+\frac{\mu}{p_2^2}\int_{-a}^{a}\mathrm{X}_4'(\eta,p)\left(\left(4\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}-p_2^2\right)K_2-4\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}-p_1^2\right)K_1-p_2^4K_2^*\right)\right)d\eta \tag{2.3}$$

Here,

$$\tilde{K}_{jk} = \frac{\partial^2}{\partial x^2}K_j + \left(p_1^2 - \frac{\partial^2}{\partial x^2}\right)K_k, \quad \hat{K}_{jk} = \left(2\frac{\partial^2}{\partial x^2} - p_2^2\right)K_j - 2\left(\frac{\partial^2}{\partial x^2} - p_1^2\right)K_k$$

$$K_j = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\alpha(\eta - x) - \gamma_j|y|}}{2\gamma_j}d\eta = -\frac{1}{2\pi}K_0(p_j\sqrt{(\eta - x)^2 + y^2})$$

$$K_j^* = \frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\alpha(\eta - x) - \gamma_j|y|}}{2\alpha\gamma_j}d\alpha; \quad \gamma_j = \sqrt{\alpha^2 + p_j^2}, \quad p_j = pc_j^{-1}; \quad j, k = 1, 2$$

$K_0(x)$ is a modified Bessel function and $X_4'(\eta, p)$ is the derivative with respect to the variable η of the Laplace transform of the discontinuity in the displacement $\chi_4(\eta,t)$. Integration by parts, taking account of formula (1.2), was carried out in the integrals containing the discontinuity in the displacements when deriving formulae (2.3).

Representation in terms of a discontinuous solution enables us to find the stresses and displacements in the matrix if their jumps are known. We shall obtain integral equations in order to find the discontinuities from the conditions on the inclusion (2.1). In order to do this, we initially differentiate the last two equalities of (2.1) with respect to the variable x and we add to the result the conditions of equivalence of the initial and differentiated equalities. We obtain

$$\begin{aligned} &V_1'(x, -0, p) = G(p) - V_0'(x, 0, p), \quad U_1'(x, -0, p) = -U_0'(x, 0, p) \\ &V_1(-a, -0, p) = A_1(p) - G(p)a - V_0(-a, 0, p), \\ &U_1(-a, -0, p) = A_2(p) - U_0(-a, 0, p) \end{aligned} \tag{2.4}$$

After substituting expressions (2.3) into the first two equalities of (2.4) and into the first equality of (2.1), we obtain a system of three integral equations in the transforms of the unknown discontinuities, the matrix-operator notation for which has the form

$$B\Phi + Q\Gamma\Phi + R\Phi = F, \quad \zeta \in [-1, 1] \tag{2.5}$$

The following notation has been introduced here

$$\Gamma\Phi = \left\langle \frac{\Phi(z, q)}{z - \zeta} \right\rangle, \quad R\Phi = \langle \Phi(z, q)R(z - \zeta, q) \rangle, \quad \langle f \rangle = \frac{1}{2\pi}\int_{-1}^{1} f dz$$

$$\Phi = [\Phi_1(z, q), \Phi_2(z, q), \Phi_3(z, q)]^T, \quad F = [f_1(\zeta), f_2(\zeta), f_3(\zeta)]^T$$

$$B = \left\| \begin{matrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{matrix} \right\|, \quad Q = \left\| \begin{matrix} -\xi^2 & 0 & 2(1 - \xi^2) \\ -\frac{1 - \xi^2}{2} & 0 & -\xi^2 \\ 0 & -\frac{1 - \varepsilon^2}{2} & 0 \end{matrix} \right\|, \quad R = \left\| \begin{matrix} T_4 & 0 & T_5 \\ T_2 & 0 & T_3 \\ 0 & T_1 & 0 \end{matrix} \right\|$$

$$f_1(\zeta) = -\frac{c_2}{a\mu}\mathrm{T}_{yx}^0\left(a\zeta, 0, \frac{c_2}{a}q\right), \quad f_2(\zeta) = -\frac{c_2}{a}V_0'\left(a\zeta, 0, \frac{c_2}{a}q\right)$$

$$f_3(\zeta) = -\frac{c_2}{a}U_0'\left(a\zeta, 0, \frac{c_2}{a}q\right), \quad z = \frac{\eta}{a}, \quad \zeta = \frac{x}{a}, \quad q = \frac{a}{c_2}p, \quad \xi^2 = \frac{c_2}{c_1}$$

$$\mathrm{X}_k\left(az, \frac{c_2}{a}q\right) = \frac{a}{c_2\mu}\Phi_k(z, q), \quad k = 1, 2, \quad \mathrm{X}_4'\left(az, \frac{c_2}{a}q\right) = \frac{a}{c_2}\Phi_3(z, q) \tag{2.6}$$

Likewise, from the remaining equalities of (2.4) we find

$$\left\langle \Phi_1(z,q)\left[\frac{1+\xi^2}{2}\ln|z+1| + T_6(q(z+1))\right]\right\rangle +$$

$$+\langle \Phi_3(z,q)[\xi^2\ln|z+1| + T_7(q(z+1))]\rangle = \alpha_1(q) - g(q) - f_{01}(q)$$

$$\left\langle \Phi_2(z,q)\left[\frac{1+\xi^2}{2}\ln|z+1| + T_8(q(z+1))\right]\right\rangle = \alpha_2(q) - f_{02}(q)$$

$$f_{01}(q) = \frac{c_2}{a^2}V_0\left(-1, \frac{c_2}{a}q\right), \quad f_{02}(q) = \frac{c_2}{a^2}U_0\left(-1, \frac{c_2}{a}q\right) \tag{2.7}$$

The following representations of modified Bessel functions was used in separating out the singular component of the kernels of the integral operators in relations (2.50 and (2.7):

$$K_0(\xi qz) = -\ln|z| + F_0(\xi qz), \quad \xi qK_1(\xi qz) = \frac{1}{z} + F_1(\xi qz)$$

$$\frac{K_1(\xi qz)}{\xi qz} = \frac{1}{\xi^2q^2z^2} + \frac{1}{2}\ln|z| + F_2(\xi qz)$$

$$\frac{K_1(\xi qz)}{(\xi qz)^2} = \frac{1}{\xi^3q^3z^3} - \frac{1}{4\xi qz} + \frac{1}{2\xi qz}\left(\ln|z| + \ln\frac{\xi q}{2} + C\right) + F_3(\xi qz)$$

$$\frac{K_0(\xi qz)}{\xi qz} = \frac{1}{qz}\left(\ln|z| + \ln\frac{\xi q}{2} + C\right) + F_4(\xi qz)$$

In these formulae $F_i(\xi qz)(i=0,\ldots,4)$ are functions which are bounded when $z \to 0$, the actual form of which is not given here because of their length, the functions $T_k(z,q)(k=1,\ldots,8)$ are linear combinations of them, and C is Euler's constant.

To Eqs. (2.5) and (2.7), it is necessary to add the condition which follows from equality (1.2):

$$\langle \Phi_3(z,q)\rangle = 0 \tag{2.8}$$

The quantities

$$\alpha_j(q) = \frac{c_2}{a^2}A_j\left(\frac{c_2}{a}q\right), \quad g(q) = \frac{c_2}{a}G\left(\frac{c_2}{a}q\right)$$

are found from the equations of motion of the inclusion (2.2) which, in the new notation, have the form

$$\alpha_j(q) = \frac{\pi\bar{\rho}}{2\varepsilon q^2}\langle \Phi_j(z,q)\rangle, \quad g(q) = \frac{3\pi\bar{\rho}}{8\varepsilon q^2}\langle \Phi_1(z,q)z\rangle; \quad \varepsilon = \frac{h}{2a}, \quad \bar{\rho} = \frac{\rho_1}{\rho_0} \tag{2.9}$$

where ρ_1 is the density of the matrix.

In order to separate out the singular component of system (2.5), we multiply both sides of equality (2.5) by the matrix Q^{-1} (this matrix exists since $\det Q \neq 0$). As a result, we obtain

$$M\Phi + E\Gamma\Phi + Q^{-1}R\Phi = Q^{-1}F \tag{2.10}$$

where $M = Q^{-1}B$, and E is the unit matrix. As was done when solving the analogous problem concerning harmonic oscillations,[3] after this we introduce the unknown functions

$$\Psi(z,q) = P^{-1}\Phi(z,q), \quad \Phi(z,q) = P\Psi(z,q), \quad \Psi = [\Psi_1, \Psi_2, \Psi_3]^T \tag{2.11}$$

The matrix p was constructed in such a manner that $P^{-1}MP = D$, where D is a third-order diagonal matrix. This matrix exists [5] and, since the eigenvalues of the matrix M are different, it can be easily constructed. In this case, the diagonal elements of the matrix D are

$$D_{11} = \lambda_1 = 0, \quad D_{22} = \lambda_2 = 1/2, \quad D_{33} = \lambda_3 = -1/2$$

After substituting the new functions into equality (2.5) and multiplying the resulting equality by P^{-1}, we arrive at the system of singular integral equations

$$\lambda_k\Psi_k(\zeta,q) + \left\langle\frac{\Psi_k(z,q)}{z-\zeta}\right\rangle + \sum_{i=1}^{3}\langle L_{ki}\Psi_i(z,q)\rangle = Y_k(\zeta,q) \tag{2.12}$$

where L_{ki} and Y_k are elements of the following matrix and vector

$$L = P^{-1}A^{-1}R(z-\zeta)P, \quad Y = P^{-1}A^{-1}F$$

It is necessary to consider system (2.12) together with equalities (2.7)–(2.9) which, after the introduction of the new unknown functions (2.11), have the form

$$\begin{aligned}&\chi_{106}(q)+\chi_{267}(q)-\chi_{367}(q) = \alpha_1(q)-g(q)+f_{01}(q)\\&\chi_{208}(q)+\chi_{308}(q) = \alpha_2(q)+f_{02}(q)\end{aligned} \tag{2.13}$$

Here,

$$\chi_{k0l}(q) = \left\langle \Psi_k(z,q)\left[\frac{1+\xi^2}{2}\ln|z+1| + T_l(q(z+1))\right]\right\rangle, \quad k = 1,2,3, \quad l = 6,8$$

$$\chi_{k67}(q) = \left\langle \Psi_k(z,q)\left[-\xi^2 T_6(q(z+1)) + \frac{1+\xi^2}{2}T_7(q(z+1))\right]\right\rangle$$

$$\langle\Psi_2(z,q)\rangle - \langle\Psi_3(z,q)\rangle = 0$$

$$\alpha_1(q) = \frac{\pi\bar{\rho}}{2\varepsilon q^2}\langle\Psi_1(z,q)-\xi^2\Psi_2(z,q)+\xi^2\Psi_3(z,q)\rangle, \ \alpha_2(q) = \frac{\pi\bar{\rho}}{2\varepsilon q^2}\langle\Psi_2(z,q)+\Psi_3(z,q)\rangle$$

$$g(q) = \frac{3\pi\bar{\rho}}{8\varepsilon q^2}\langle(\Psi_1(z,q)-\xi^2\Psi_2(z,q)+\xi^2\Psi_3(z,q))z\rangle$$

The solution of system (2.12), which is integrable when $\zeta\in[-1,1]$, has the form [6]

$$\begin{aligned}&\Psi_k(\zeta,q) = (1-\zeta)^{\alpha_k}(1+\zeta)^{\beta_k}W_k(\zeta,q), \quad k = 1,2,3\\&\alpha_1 = \beta_1 = -1/2, \quad \alpha_2 = \beta_3 = -1/4, \quad \alpha_3 = \beta_2 = -3/4\end{aligned} \tag{2.14}$$

With respect to the functions $W_k(\zeta,q)$, we assume that they satisfy the Hölder condition when $\zeta\in[-1,1]$ and we approximate them with the interpolating polynomials [6]

$$W_k(\zeta,q) = \sum_{m=1}^{n}\frac{W_k(z_{km},q)P_n^{\alpha_k,\beta_k}(\zeta)}{(\zeta-z_{km})[P_n^{\alpha_k,\beta_k}(z_{km})]'}, \quad k = 1,2,3 \tag{2.15}$$

where z_{km} are the roots of the Jacobi polynomial $P_n^{\alpha_k,\beta_k}(\zeta)$.

We will now solve the problem of determining the values of the function $W_k(\zeta_{km},q)$ at the interpolation points from system (2.12). For this purpose, we consider the singular integral operators

$$S_1[\Psi_1] = \left\langle\frac{\Psi_1(z,q)}{z-\zeta}\right\rangle, \quad S_k[\Psi_k] = \Psi_k(\zeta,q)+2(-1)^{k+1}\left\langle\frac{\Psi_k(z,q)}{z-\zeta}\right\rangle, \quad k = 2,3$$

and introduce the notation

$$P_{kj} = \frac{1}{2\pi}\sum_{m=1}^{n}\frac{A_m^{(k)}W_k(z_{km},q)}{z_{km}-\zeta_{kj}}, \quad A_m^{(1)} = \frac{\pi}{n}, \quad A_m^{(k)} = \frac{\pi P_{n-1}^{-\alpha_k,-\beta_k}(z_{km})}{\sqrt{2}[P_n^{\alpha_k,\beta_k}(z_{km})]'}$$

$$j = 1,2,\ldots,n-1; \ k = 1,2,3$$

The quadrature formulae

$$S_1[\Psi_1(\zeta_{1j},q)] = P_{1j}, \quad S_k[\Psi_k(\zeta_{kj},q)] = 2(-1)^{k+1}P_{kj} \tag{2.16}$$

hold, where ζ_{kj} are the roots of the polynomial $P_{n-1}^{-\alpha_k,-\beta_k}(\zeta)(j = 1,2,\ldots,n-1; \ k = 1,2,3)$.

The integrals with a logarithmic singularity can be evaluated using the quadrature formula [7]

$$2\pi\langle \ln(1+z)\Psi_k(z,q)\rangle = \sum_{m=1}^{n} A_m^{(k)} W_k(z_{km},q) D_m^{(k)}$$

$$D_m^{(k)} = 2\ln 2 + (-1)^k 2\pi + 2(\psi(1+\alpha_k) - C) - 2\sum_{m=1}^{n-1} (-1)^m \frac{P_m^{\alpha_k,\beta_k}(z_{km})(m-1)!}{(-\alpha_k)_m} \tag{2.17}$$

where $(-\alpha_k)_m$ is the Pochhammer symbol.

In relations (2.12) and (2.13), we approximate the singular integrals by sums using formulae (2.16) and (2.17) and the regular integrals using Gauss–Jacobi quadrature formulae.[8] As a result, relations (2.12) and (2.13) are replaced by the following system of linear algebraic equations in the transforms of the unknown functions at the points of interpolation

$$P_{kj} + \sum_{i=1}^{3} Q_{kji} = Y_k(\zeta_{kj}, q), \quad j = 1, \dots, n-1; \quad k = 1, 2, 3$$

$$Z_{106}(q) + Z_{267}(q) - Z_{367}(q) = f_{01}(q) + \alpha_1(a) - g(q)$$

$$Z_{208}(q) + Z_{308}(q) = f_{02}(q) + \alpha_2(q)$$

$$N_2^{(0)}(q) - N_3^{(0)}(q) = 0$$

$$\alpha_1(q) = N_1^{(0)}(q) - \xi^2 N_2^{(0)}(q) + \xi^2 N_3^{(0)}(q)$$

$$\alpha_2(q) = N_1^{(0)}(q) + N_2^{(0)}(q)$$

$$g(q) = \frac{3}{4}(N_1^{(1)}(q) - \xi^2 N_2^{(1)}(q) + \xi^2 N_3^{(1)}(q))$$

$$Q_{kji} = \frac{1}{2\pi}\sum_{m=1}^{n} A_m^{(i)} W_k(z_{im}, q) L_{ki}(z_{im} - \zeta_{kj})$$

$$N_k^{(j)}(q) = \frac{\bar{\rho}}{4\varepsilon q^2}\sum_{m=1}^{n} A_m^{(k)} W_k(z_{km}, q)(z_{km})^j, \quad j = 0, 1$$

$$Z_{k0l}(q) = \frac{1}{2\pi}\sum_{m=1}^{n} A_m^{(k)} W_k(z_{km}, q)\left(\frac{1+\xi^2}{2} D_m^{(k)} + T_l(q(z_{km}+1))\right), \quad l = 6, 8$$

$$Z_{k67}(q) = \frac{1}{2\pi}\sum_{m=1}^{n} A_m^{(k)} W_k(z_{km}, q)\left(-\xi^2 T_6(q(z_{km}+1)) + \frac{1+\xi^2}{2} T_7(q(z_{km}+1))\right) \tag{2.18}$$

After solving system (2.18), the approximate values of the unknown functions can be found using formulae (2.14) and (2.15).

As previously,[2] we shall consider the coefficients accompanying the step singularities in the stresses

$$k_j^{\pm}(t) = \lim_{x \to \pm a \mp 0} \chi_j(x, t)(a \mp x)^{3/4}, \quad j = 1, 2$$

as a quantity which characterizes the concentration of elastic stresses close to the inclusion.

We shall henceforth refer to them as the stress intensity factors (SIFs) for the delaminated inclusion. If the notation (2.6) is used, the last formula becomes

$$k_j^{\pm}(\tau) = a^{3/4}\mu K_j^{\pm}(\tau), \quad K_j^{\pm}(\tau) = \lim_{\zeta \to \pm 1 \mp 0}(1 \mp \zeta)^{3/4}\varphi_j(\zeta, \tau), \quad \tau = \frac{c_2}{a}t \tag{2.19}$$

In the case of the Laplace transforms of the functions $K_j^{\pm}(\tau)$

$$\bar{K}_j^{\pm}(q) = \lim_{\zeta \to \pm 1 \mp 0}(1 \mp \zeta)^{3/4}\Phi_j(\zeta, q) \tag{2.20}$$

after substituting relations (2.14) and (2.15) into formula (2.2) and taking the limit, we finally obtain

$$\overline{K}_1^{\pm}(q) = \xi^2 N^{\pm}(q), \quad \overline{K}_2^{\pm}(q) = N^{\pm}(q)$$
$$N^{+}(q) = 2^{-1/4}W_3(1,q), \quad N^{-}(q) = 2^{-1/4}W_2(-1,q) \tag{2.21}$$

After the solution of system (2.12) has been found, formulae (2.21) and the last three equations of (2.18) enable as to find the transforms of the SIFs and the amplitudes of the oscillations of the inclusion. It is found that the SIFs are proportional to the same quantities $N^{\pm}$ by means of which the stress concentration close to the inclusion will also be determined. Their inverse transforms $n^{\pm}(\tau)$ were re-established using numerical methods by replacing a Mellin integral by a Fourier series.[9,10]

3. Results of numerical investigations

We will now consider the results of numerical investigations of the relations $n^{+}(\tau)$ when longitudinal and transverse waves act on the inclusion, which are respectively specified by potentials $\varphi_0 (j=1)$ and $\Psi_0 (j=2)$ of the form

$$\sigma_j^2 H(\sigma_j) \tag{3.1}$$

or

$$\frac{c_j}{\omega}\cos\left(\frac{\omega}{c_j}\sigma_j\right)H(\sigma_j) \tag{3.2}$$

where

$$\sigma_j = c_j t - (x+a)\cos\theta_0 - y\sin\theta_0$$

and $H(t)$ is the Heaviside function. It was assumed in the calculations that the dimensionless frequency $f = a\omega c_2^{-1} = 3$

Graphs of the function $n^{+}(\tau)$ in the case of the action of shock waves with potentials of the form of (3.1) on an inclusion are shown in Fig. 1: a longitudinal wave with an angle of propagation $\theta_0 = 0$ and a transverse wave with an angle of propagation $\theta_0 = \pi/2$. In both cases, a sharp increase in the absolute values of $n^{+}(\tau)$ up to the extremal values is observed at the instant when the wave arrives followed by an oscillating trend to a certain constant value. The extremal values of $n^{+}(\tau)$ and the time required to reach a constant value are greater the greater the relative density of the inclusion $\bar{\rho} = \rho_1/\rho_0$.

The relations $n^{+}(\tau)$, resulting from the action of the analogous harmonic waves with potentials of the form of (3.2) on the inclusion, are shown in Fig. 2. For all values of the relative density of the inclusion, the oscillations reach steady conditions corresponding to harmonic oscillations, and the time required to reach steady conditions increases as the relative density of the inclusion becomes greater. In passing

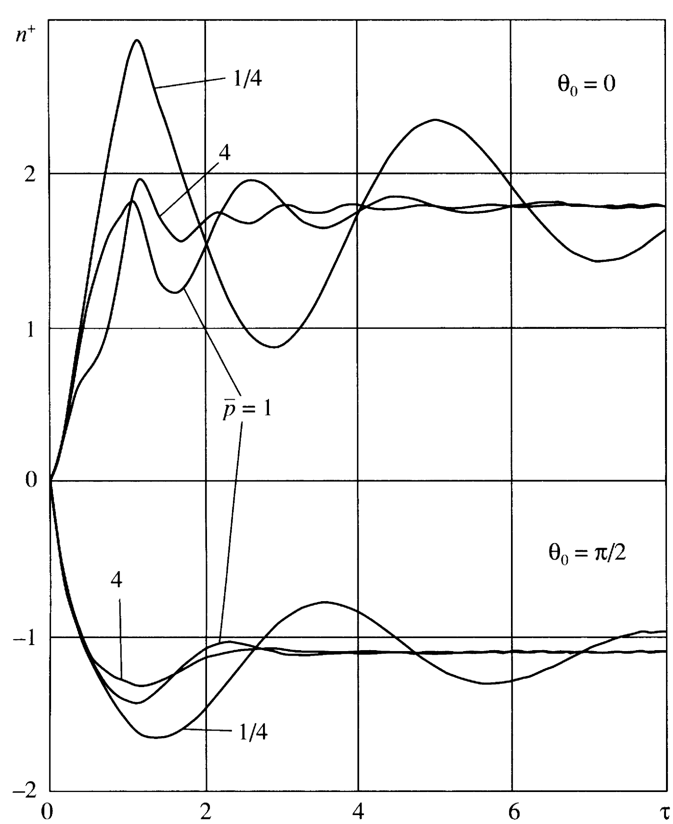

Fig. 1.

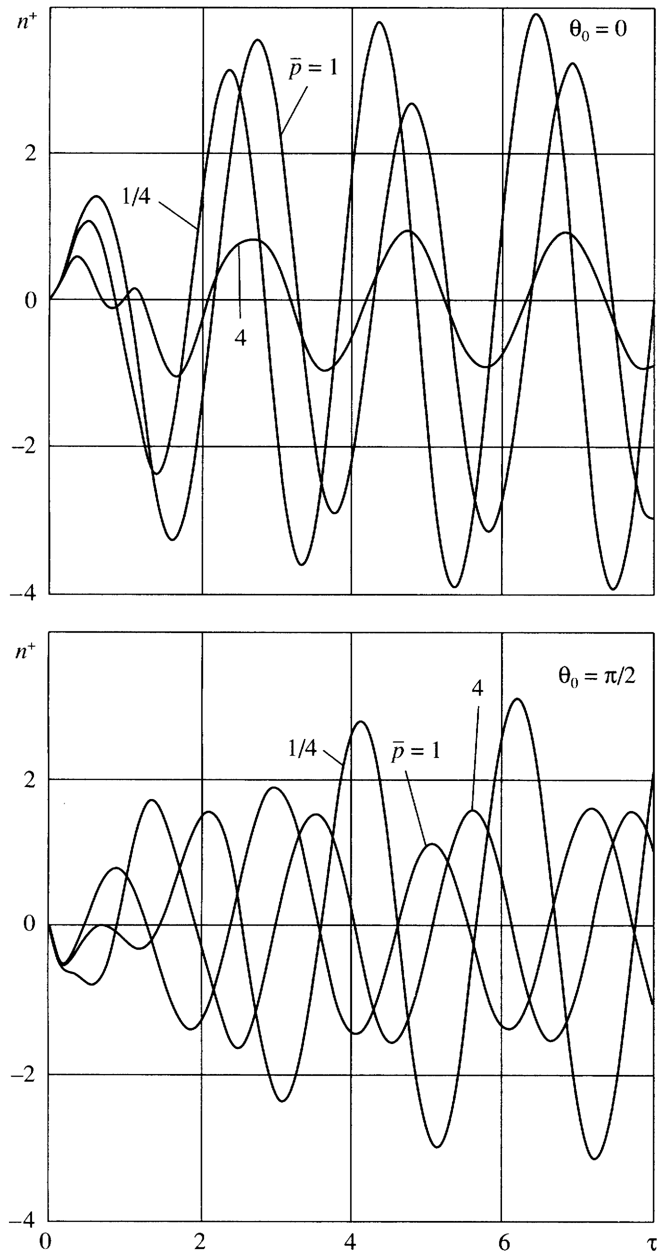

Fig. 2.

to the steady condition for harmonic oscillations $0 < \pi < 2$, no values of $n^+(\tau)$, are observed which exceed the maximum values for the state which has been established. This fact only enables one to solve the steady problem when investigating the stress state near such an inclusion.

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